The Yang-Baxter equation

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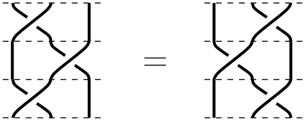
Let V be a vector space. We want to find $c \in \mathbf{GL}(V \otimes V)$ that are solutions to the Yang–Baxter equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$



Let us explain the notation.

- If V be a vector space of dimension n, say with basis $\{v_1, \ldots, v_n\}$, then the tensor product $V \otimes V$ is the n^2-dimensional vector space with basis

$$\{v_i \otimes v_j : 1 \leq i, j \leq n\}.$$

- If $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$, then the tensor product (or Kronecker product) of A and B is the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Concrete example:
If V is the two-dimensional vector space with basis $\{x, y\}$, then $V \otimes V$ is a four-dimensional vector space with basis

$$\{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}.$$

Let us compute a Kronecker product:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{pmatrix}.$$

Some solutions:

- The flip: $\tau(x \otimes y) = y \otimes x$.
- The matrix

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

satisfies

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Another example:
Let V with basis $\{x, y\}$ and $q \in \mathbb{C} \setminus \{0\}$. Then

$$\begin{aligned} c(x \otimes x) &= x \otimes x, \\ c(y \otimes y) &= y \otimes y, \\ c(x \otimes y) &= q(y \otimes x), \\ c(y \otimes x) &= q(x \otimes y) + (1 - q^2)(y \otimes x), \end{aligned}$$

is a solution.

Remark:
This example comes from the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ and explains the Jones polynomial of a knot!

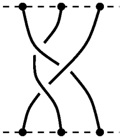
Important fact:
Solutions of the Yang–Baxter equation produce representations of the Braid Group.

Braid groups were introduced by Emil Artin in 1925, although they were already implicit in Adolf Hurwitz's work in 1891.

Braid groups find applications in topology and knot theory, cryptography, quantum computers, etc.

Before defining the Braid Group $\mathbb{B}_n$ for $n \in \{1, 2, 3, \dots\}$, let us start with an example.

This is a typical element of $\mathbb{B}_3$:



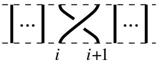
The Braid Group $\mathbb{B}_n$ has generators

$$\sigma_1, \sigma_2, \cdots, \sigma_{n-1},$$

and relations

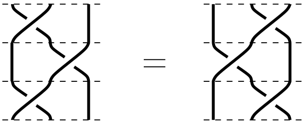
$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \quad 1 \leq i \leq n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i & \quad |i-j| \geq 2. \end{aligned}$$

If each σ_i is written as

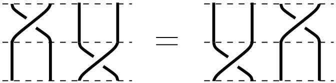


then

- the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is



- and the relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ is



Braid Group representations:
If c is a solution of the Yang–Baxter equation, let

$$c_i = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} \in \mathbf{GL}(V^{\otimes n}).$$

Then

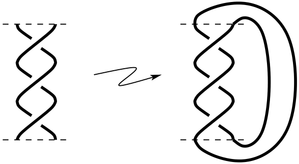
- $c_1, \cdots, c_{n-1}$ satisfy the relations of the Braid Group.
- The map

  $$\rho_n : \mathbb{B}_n \to \mathbf{GL}(V^{\otimes n}), \quad \rho_n(\sigma_i) = c_i,$$

  is a group homomorphism.

Why the Braid Group?

Every knot can be presented as the closure of a braiding. This is a well-known theorem of Alexander.

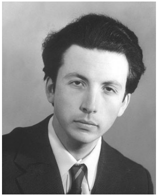


Problem

Construct solutions of the Yang–Baxter equation.

Attempts to find non-trivial solutions lead to the theory of quantum groups.

In 1992 Drinfeld propose to study set-theoretical solutions of the Yang–Baxter equation.



Vladimir Drinfeld.

A set-theoretical solution is a pair (X, r), where X is a set and $r\colon X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r).$$

Set-theoretical solutions produce solutions!

Examples:

- The flip: $r(x, y) = (y, x)$.
- Let X be a set and $f, g\colon X \to X$ be bijections such that $fg = gf$. Then
$$r(x, y) = (f(y), g(x))$$
is a solution.
- Let $X = \mathbb{Z}/n$. Then
$$r(x, y) = (2x - y, x)$$
is a solution.

More examples:

Let X be a group.

- $r(x, y) = (xyx^{-1}, x)$ is a solution.
- $r(x, y) = (xy^{-1}x^{-1}, xy^2)$ is a solution.

Problem

Construct (finite) set-theoretical solutions.

The first papers devoted to set-theoretical solutions are those of Etingof, Schedler and Soloviev[1] and Gateva-Ivanova and Van den Bergh[2].

Both papers deal with non-degenerate involutive solutions, i.e. solutions $r\colon X \times X \to X \times X$ such that $r^2 = \mathrm{id}$ and

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

where σ_x and τ_x are permutations of X for each $x \in X$.

[1] Duke Math. J. 100 (1999), no. 2, 169–209.
[2] J. Algebra 206 (1998), no. 1, 97–112.

Number of finite non-degenerate involutive solutions.

size	1	2	3	4	5	6	7	8	9
solutions	1	2	5	23	88	595	3456	34528	?

Not much is known about solutions. An important conjecture was the following:

Conjecture (Gateva-Ivanova, 2004)

Let (X, r) be a non-degenerate involutive solution with X finite. Assume that $r(x, x) = (x, x)$ for all $x \in X$. Then $\sigma_x = \sigma_y$ for some $x \neq y$.

Cedó, Jespers y Okniński proved[3] that the conjecture is true in the case where the group

$$L(X, r) = \langle \sigma_x : x \in X \rangle$$

is abelian.

[3] Adv. Math. 224 (2010), no. 6, 2472–2484.

Gateva-Ivanova conjecture is not true. I constructed an infinite family of counterexamples[4].

Example: Let $X = \{1, \ldots, 8\}$ and

$$r(x, y) = (\varphi_x(y), \varphi_y(x)),$$

where

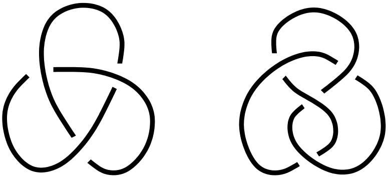
$$\begin{aligned} \varphi_1 &= (57), & \varphi_2 &= (68),\\ \varphi_3 &= (26)(48)(57), & \varphi_4 &= (15)(37)(68),\\ \varphi_5 &= (13), & \varphi_6 &= (24),\\ \varphi_7 &= (13)(26)(48), & \varphi_8 &= (15)(24)(37). \end{aligned}$$

Then (X, r) is a counterexample to Gateva-Ivanova conjecture.

[4] J. Pure Appl. Algebra 220 (2016), no. 5, 2064–2076.

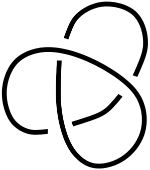
Let us see an application of finite solutions to the theory of knots.

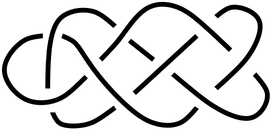
A knot is a simple closed polygonal in $\mathbb{R}^3$.



In practice, when drawing knots it is assumed that are are so many straight lines segments that the curves appear well rounded.

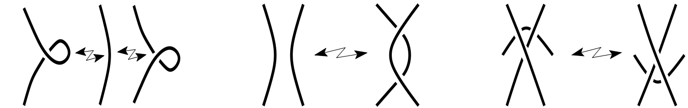
Are there non-trivial knots?
Sometimes trivial knots do not look trivial at all!





Two trivial knots.

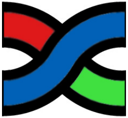
Reidemeister theorem states that two knots are equivalent if and only if they are connected by a finite number of certain transformations:



These transformations are known as Reidemeister moves.

To check whether a knot is trivial or not, one can try to use the 3-coloring invariant. What does it mean?

A knot can be colored with **three colors** if each arc of the diagram (from one under-pass to the next) can be colored red, blue or green so that all three colors are used and at each crossing either one color or all three colors appear.



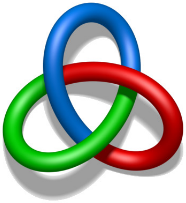
A crossing with three colors.

Theorem

If a knot admits a three coloring, then it is non-trivial.

Proof (exercise): One needs to check that three-colorability is unchanged by Reidemeister moves. Since the trivial knot does not admit a three-coloring, the claim follows.

Application:
The trefoil is a non-trivial knot:



Certain solutions of the Yang–Baxter equation are far reaching generalizations of the three-coloring!

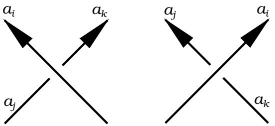
Let (X, r) be a solution, where

$$r(x, y) = (f(x, y), x), \quad f\colon X \times X \to X.$$

Assume that your knot is oriented. Label the arcs of the diagram with the elements of X in such a way that

$$a_k = f(a_i, a_j),$$

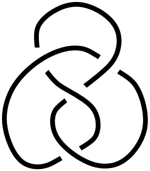
where



So coloring a knot with three colors is just to label the arcs with the elements of the solution (X, r), where $X = \mathbb{Z}/3$ and

$$r(x, y) = (2x - y, x).$$

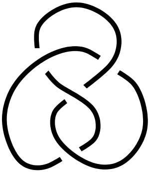
Exercise:
Prove that the following knot is non-trivial:



Solution 1:
Use the solution (X, r), where $X = \mathbb{Z}/5$ and

$$r(x, y) = (2x - y, x)$$

Exercise:
Prove that the following knot is non-trivial:



Solution 2:
Use the solution (X, r), where

$$\left\{ \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathbf{SL}(2, 3).$$

and

$$r(x, y) = (xyx^{-1}, x).$$

Thanks!