Comparisons between polynomial norms and a little bit of random polynomials

Daniel E. Galicer[1]

[1]Universidad de Buenos Aires and CONICET

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The basics

Examples:

- $P_1(z_1, z_2, z_3) = 3z_1^2 - 2z_1z_2 + 7z_2^2 - 5z_3^2 + 4z_1z_3.$

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- $P_1(z_1,z_2,z_3) = 3z_1^2 - 2z_1z_2 + 7z_2^2 - 5z_3^2 + 4z_1z_3.$
- $P_2(z_1,z_2,z_3,z_4) = 5z_1^5 + \frac{3}{2}z_1z_2^4 - z_2^3z_3z_4 + 2z_3z_4^4.$

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In both cases, all the terms have the same degree ($\equiv$ homogeneity degree).

The basics

Definition

An m-homogeneous polynomial in n complex variables is a function

$$P : \mathbb{C}^n \longrightarrow \mathbb{C},$$

that can be written as

$$P(z) = \sum_{\mathbf{j} \in \mathscr{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}},$$

where $\mathscr{J}(m,n) := \{\mathbf{j} = (j_1, \ldots, j_m) \in \{1, \ldots, n\}^m : j_1 \leq j_2 \leq \cdots \leq j_m\}$, $c_{\mathbf{j}}(P) \in \mathbb{C}$ and $z_{\mathbf{j}} = z_{j_1} \cdots z_{j_m}$.

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- $P(\lambda z_1, \ldots, \lambda z_n) = \lambda^m P(z_1, \ldots, z_n)$.

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- $P(\lambda z_1, \dots, \lambda z_n) = \lambda^m P(z_1, \dots, z_n)$.
- The elements $(z_{\mathbf{j}})_{\mathbf{j} \in \mathscr{J}(m,n)}$ are called the *monomials*.

The basics

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- The monomials $(z_{\mathbf{j}})_{\mathbf{j}\in\mathscr{J}(m,n)}$ form a basis of the space $\mathscr{P}(^m\mathbb{C}^n)$, and therefore

$$\dim \mathscr{P}(^m\mathbb{C}^n) = |\mathscr{J}(m,n)| = \binom{n+m-1}{m} \sim n^m.$$

Norms on $\mathscr{P}({}^m\mathbb{C}^n)$

- ℓ_p^n stands for $\mathbb{C}^n$ endowed with the norm

$$\|z\|_{\ell_p^n} := \left(\sum_{j=1}^{n} |z_j|^p \right)^{1/p} \text{ if } p < \infty,$$

and

$$\|z\|_{\ell_\infty^n} := \max_j |z_j|.$$

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Uniform/sup norm:

For $1 \leq p \leq \infty$,

$$\|P\|_{\mathscr{P}(^m\ell_p^n)} := \sup_{z \in B_{\ell_p^n}} |P(z)|.$$

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For $1 \leq r < \infty$,

$$|P|_r := \Big(\sum_{\mathbf{j} \in \mathscr{J}(m,n)} |c_{\mathbf{j}}(P)|^r \Big)^{\frac{1}{r}},$$

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Question

How do they relate each other?

The problem...

More precisely,

Let $A^m_{p,r}(n)$ and $B^m_{r,p}(n)$ be the smallest constants that fulfill the following inequalities: for every m-homogeneous polynomial P in n complex variables,

$$|P|_r \leq A^m_{p,r}(n)\, \|P\|_{\mathscr{P}({}^m\ell_p^n)}, \qquad \|P\|_{\mathscr{P}({}^m\ell_p^n)} \leq B^m_{r,p}(n)\, |P|_r.$$

How these constants behave in terms of the number of variables n? Which is their exact asymptotic growth?

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Essentially, we want to relate the summability of the coefficients of a given homogeneous polynomial with its uniform norm for ℓ_p-spaces.

The easy part... $B^m_{r,p}(n)$

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leq Cb_n$ for every n and we denote $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$.

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Proposition

We have

$$B^m_{r,p}(n) \sim \begin{cases} 1 & \text{for } r \leq p', \\ n^{m(1-\frac{1}{p}-\frac{1}{r})} & \text{for } r \geq p'. \end{cases}$$

The easy part... $B^m_{r,p}(n)$

Suppose first that $r \leq p'$. Then

$$\|P\|_{\mathscr{P}(^m\ell_p^n)} = \sup_{z \in B_{\ell_p^n}} \Big| \sum_{\mathbf{j} \in \mathscr{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}} \Big|$$

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&\leq \sup_{z\in B_{\ell_p^n}} |P|_{p'} (\sum_{k=1}^n |z_k|^p)^{\frac{m}{p}}
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\end{aligned}$$

$$n^{m(1-\frac{1}{p}-\frac{1}{r})} = \frac{n^{m(1-\frac{1}{p})}}{n^{\frac{m}{r}}} \ll \frac{\|P\|_{\mathcal{P}(^m\ell_p^n)}}{|P|_r} \leq B^m_{r,p}(n).$$

The real job... $A^m_{p,r}(n)$

- $A^m_{p,r}(n) \equiv$ the smallest constant such that for every m-homogeneous polynomial P in n complex variables,

$$|P|_r \leq A^m_{p,r}(n)\ \|P\|_{\mathscr{P}(^m\ell_p^n)}.$$

Some known inequalities... Bohnenblust and E. Hille (31'), Hardy and J. Littlewood ('34), Praciano-Pereira (81'), Dimant-Sevilla Peris (2013).

$$\begin{array}{lrl} (i) & A^m_{p,\frac{2m}{m+1}}(n) \sim 1 & \text{for } p = \infty, \\ (ii) & A^m_{p,\frac{2mp}{mp+p-2m}}(n) \sim 1 & \text{for } 2m \leq p < \infty, \\ (iii) & A^m_{p,\frac{p}{p-m}}(n) \sim 1 & \text{for } m \leq p \leq 2m. \end{array}$$

The real job... $A^m_{p,r}(n)$

G., Mansilla, Muro (2016)

$$
\begin{cases}
A^m_{p,r}(n) \sim 1 & \text{for } (A): [\frac{1}{2} \le \frac{1}{r} \le \frac{m+1}{2m} - \frac{1}{p}] \text{ or } [\frac{1}{r} \le \frac{1}{2} \wedge \frac{m}{p} \le 1 - \frac{1}{r}], \\
A^m_{p,r}(n) \sim n^{\frac{m}{p}+\frac{1}{r}-1} & \text{for } (B): [\frac{1}{2m} \le \frac{1}{p} \le \frac{1}{m} \wedge -\frac{m}{p} + 1 \le \frac{1}{r}], \\
A^m_{p,r}(n) \sim n^{m(\frac{1}{p}+\frac{1}{r}-\frac{1}{2})-\frac{1}{2}} & \text{for } (C): [\frac{m+1}{2m} \le \frac{1}{r} \wedge \frac{1}{p} \le \frac{1}{2}] \text{ or} \\
 & \quad [\frac{1}{2} \le \frac{1}{r} \le \frac{m+1}{2m} \le \frac{1}{p} + \frac{1}{r} \wedge \frac{1}{p} \le \frac{1}{2}], \\
A^m_{p,r}(n) \sim n^{\frac{m}{r}+\frac{1}{p}-1} & \text{for } (D): [\frac{1}{2} \le \frac{1}{p} \wedge 1 - \frac{1}{p} \le \frac{1}{r}], \\
A^m_{p,r}(n) \ll n^{\frac{m-1}{r}} & \text{for } (E): [\frac{1}{2} \le \frac{1}{p} \le 1 - \frac{1}{r}], \\
A^m_{p,r}(n) \sim n^{\frac{1}{r}} & \text{for } (F): [\frac{m-1}{p} \le 1 - \frac{1}{r} \wedge \frac{1}{m} \le \frac{1}{p} \le \frac{1}{m-1}].
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Moreover, the power of n in (E) cannot be improved.

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\end{cases}
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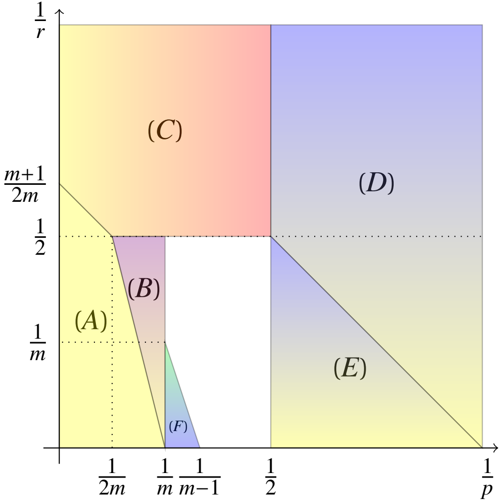
Moreover, the power of n in (E) cannot be improved.

WHAT THE FU#& IS ALL THAT?

The real job... $A^m_{p,r}(n)$



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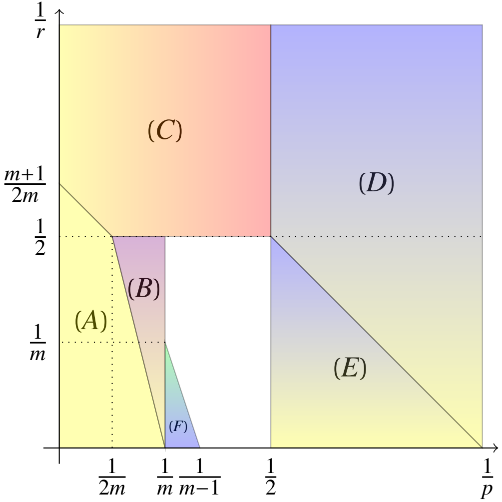

Note that for $m = 2$ the square is filled.

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• Upper bounds $\rightsquigarrow$ Classical inequalities, interpolation, properties of the associated multilinear form.

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But... How?

Idea:

Use the probabilistic method to find polynomials with small norm and many non-zero coefficients).

Randomness and polynomials

Let $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ be a family of independent Bernoulli random variables, i.e., $\varepsilon_{\mathbf{j}}:\Omega\to\{-1,1\}$ and $\mathbb{P}(\varepsilon_{\mathbf{j}}=1)=\mathbb{P}(\varepsilon_{\mathbf{j}}=-1)=\frac{1}{2}$.

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Let $\Lambda\subset\mathcal{J}(m,n)$ be a fixed index set. Consider the random polynomial

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$$\mathbb{E}\Big(\,\|P\|_{\mathcal{P}(^m\ell_\infty^n)}\Big)\ll\sqrt{n|\Lambda|}.$$

In particular, if $\Lambda=\mathcal{J}(m,n)$ then

$$\mathbb{E}\Big(\,\|P\|_{\mathcal{P}(^m\ell_\infty^n)}\Big)\ll\sqrt{n|\mathcal{J}(m,n)|}\sim n^{\frac{m+1}{2}}.$$

Randomness and polynomials

Generalizations to ℓ_p^n:

Let $P(z) := \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega) z_{\mathbf{j}}$

Boas (2000) - Bayart (2010)

$$\mathbb{E}\Big(\|P\|_{\mathcal{P}(^m \ell_p^n)} \Big) \ll \begin{cases} n^{1-\frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

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It is unlikely that that we can have a result with "good" bounds in terms of the cardinality of the monomials involved.

Randomness and polynomials

Why the previous results are important to us?

Randomness and polynomials

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For example, for $1 \leq p \leq 2$, since $\mathbb{E}\left(\|P\|_{\mathcal{P}(^m \ell_p^n)} \right) \ll n^{1-\frac{1}{p}}$ we know there are signs $(\theta_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}$ such that the polynomial $P(z) := \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \theta_{\mathbf{j}} z_{\mathbf{j}}$ verifies:

$$\|P\|_{\mathcal{P}(^m \ell_p^n)} \ll n^{1-\frac{1}{p}} \quad \text{and} \quad |P|_r = |\mathcal{J}(m,n)|^{1/r} \sim n^{m/r}.$$

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Thus,

$$n^{m(\frac{1}{r}+\frac{1}{p})-1} = \frac{n^{m/r}}{n^{1-\frac{1}{p}}} \ll \frac{|P|_r}{\|P\|_{\mathscr{P}(^m\ell_p^n)}} \le A_{p,r}^m(n)$$

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$$n^{\frac{m}{r}+\frac{1}{p}-1} \ll A_{p,r}^m(n) \leq n^{\frac{m-1}{r}},$$

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Conclusion: we do not have a good bound or we need to find "new" extreme polynomials.

A little bit of combinatorics

Partial Steiner systems $\equiv S_p(t,m,n)$

Collection of subsets $S \subseteq \{1,\ldots,n\}$ with $|S| = m$ such that $\forall A \subseteq \{1,\ldots,n\}$ with $|A| = t$ the exists **at most** one S verifying $A \subseteq S$.

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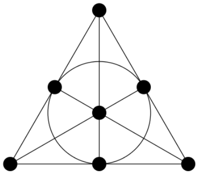

Fano's plane $\equiv S_p(2,3,7)$.

A little bit of combinatorics

Rödl (1985)

There are partial partial Steiner systems $S_p(t,m,n)$ of cardinality

$$(1-o(1))\frac{\binom{n}{t}}{\binom{m}{t}}.$$

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We will work with partial Steiner systems $S_p(m-1,m,n)$. Therefore, there are systems $S_p(k-1,k,n)$ of cardinality

$$(1-o(1))\frac{\binom{n}{m-1}}{\binom{m}{m-1}} \gg n^{m-1}.$$

Steiner polynomials

G., Muro, Sevilla (2015)

Let $m \geq 2$ and $\mathscr{S}$ be an $S_p(m-1,m,n)$ partial Steiner system of cardinality $\gg n^{m-1}$. There exist signs $(\theta_{\mathbf{j}})_{\mathbf{j}\in\mathscr{S}}$ such that the m-homogeneous polynomial $P = \sum_{\mathbf{j}\in\mathscr{S}} \theta_{\mathbf{j}} z_{\mathbf{j}}$ satisfies

$$\|P\|_{\mathscr{P}(^m\ell_p^n)} \ll \begin{cases} \log^{\frac{3p-3}{p}}(n) & \text{for } 1 \leq p \leq 2, \\ \log^{\frac{3}{p}}(n) n^{m(\frac{1}{2}-\frac{1}{p})} & \text{for } 2 \leq p < \infty. \end{cases}$$

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Thus, in (E) we have

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Random processes and polynomials

How we used this combinatorial configuration? How we can find these polynomials?

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$L_\psi = L_\psi(\Omega,\Sigma,\mathbb{P})$ is the space of all real random variables Y for which there is a constant $c>0$ verifying $\mathbb{E}(\psi(|Y|/c))<\infty$.

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$$\|Z\|_{L_\psi} = \inf\{c>0 : \mathbb{E}(\psi(|Y|/c)) \leq 1\}.$$

Random processes and polynomials

Theorem

Let $Y = (Y_x)_{x \in X}$ be a random process indexed by a metric space (X,d) in L_ψ such that, for every $x,x' \in X$,

$$\|Y_x - Y_{x'}\|_{L_\psi} \le d(x,x')$$

then

$$\mathbb{E}\Big(\sup_{x,x' \in X} |Y_x - Y_{x'}|\Big) \le 8 \int_0^{diam(X)} \psi^{-1}(N(X,d;\varepsilon))d\varepsilon \,.$$

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Idea: Given a partial Steiner system $S_p(m-1, m.n)$ define a random process $Y_z := P(z) = \sum_{\mathbf{j} \in \mathscr{S}} \varepsilon_{\mathbf{j}}(\omega) z_{\mathbf{j}}$, indexed by the ball $B_{\ell_p^n}$ and try to use the theorem.

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Let $\mathscr{S}$ be a partial Steiner system $S_p(m-1, m.n)$. For $z \in B_{\ell_2^n}$ we define the following Bernoulli process indexed by $B_{\ell_2^n}$ as

$$Y_z = \frac{1}{m} \sum_{\mathbf{j} \in \mathscr{S}} \varepsilon_{\mathbf{j}}(\omega) z_{\mathbf{j}}. \tag{1}$$

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Lemma

The Rademacher process defined in (1) *fulfils the following Lipschitz condition:*

$$\|Y_z - Y_{z'}\|_{L_{\psi_2}} \leq C \|z - z'\|_\infty,$$

for some universal constant $C \geq 1$ and every $z, z' \in B_{\ell_2^n}$.

Random processes and polynomials

As a consequence of Khintchine inequalities, the ψ_2-norm of a Bernoulli process is comparable to its L_2-norm. Now,

$$\|Y_z - Y_{z'}\|_{L_2} = \frac{1}{m}\Big(\int_\Omega |\sum_{\mathbf{j}\in\mathscr{S}} \varepsilon_{\mathbf{j}}(\omega)(z_{\mathbf{j}} - z'_{\mathbf{j}})|^2 d\mathbb{P}(\omega)\Big)^{1/2}$$

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Since $\mathscr{S}$ is an $S_p(m-1,m,n)$ partial Steiner system, given $j_1,\dots,j_{u-1},j_{u+1},\dots,j_m$ for a fixed u, there is at most one index j_u such that $(j_1,\dots,j_m)$ belongs to $\mathscr{S}$.

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Thus, $\sum_{\mathbf{j} \in \mathscr{S}} |z_{j_1} \dots z_{j_{u-1}} z'_{j_{u+1}} \dots z'_{j_m}|^2$ can be bounded by

$$\Big(\sum_{l_1=1}^{n} |z_{l_1}|^2\Big) \cdots \Big(\sum_{l_{u-1}=1}^{n} |z_{l_{u-1}}|^2\Big)\Big(\sum_{l_{u+1}=1}^{n} |z'_{l_{u+1}}|^2\Big) \cdots \Big(\sum_{l_m=1}^{n} |z'_{l_m}|^2\Big),$$

and this is less than or equal to one (since $z, z' \in B_{\ell_2^n}$).

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How do we obtain the bounds for other values of $p \neq 2$?

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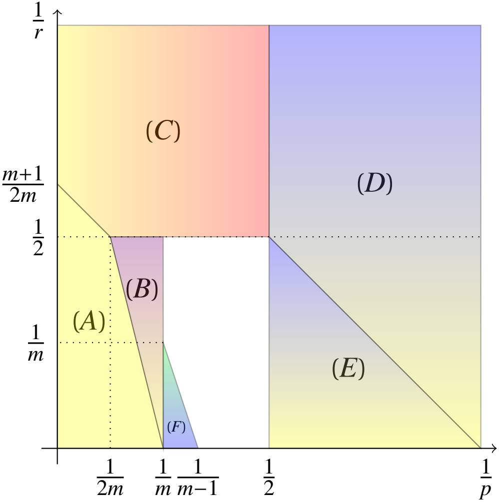
Let $m \geq 2$ and $\mathscr{S}$ be an $S_p(m-1, m, n)$ partial Steiner system of cardinality $\gg n^{m-1}$ and let $P = \sum_{\mathbf{j} \in \mathscr{S}} \varepsilon_{\mathbf{j}}(\omega) z_{\mathbf{j}}$ then

$$\mathbb{E}(\|P\|_{\mathscr{P}(^m \ell_2^n)}) \ll \log(n)^{3/2}.$$

How do we obtain the bounds for other values of $p \neq 2$? $\rightsquigarrow$ obtain bounds for $p = 1$ and $p = \infty$ (KSZ) and interpolate.

Lets go back to the graphic...



The blank region

Given a compatible couple (X,Y) of Banach spaces and $0<\theta<1$ we denote by $[X,Y]_\theta$ the intermediate space in the complex interpolation sense.

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Let $2 \le p,q \le \infty$, $0<\theta<1$ and $n,m \in \mathbb{N}$.

Relaxed version of the polynomial interpolation problem

Is the norm of the natural identity

$$\mathscr{P}({}^m[\ell_p^n,\ell_q^n]_\theta) \simeq [\mathscr{P}({}^m\ell_p^n),\mathscr{P}({}^m\ell_q^n)]_\theta,$$

independent of n?

For $m=2$ the answer is affirmative.

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Assuming a positive answer to the PIP we are able to complete all the remaining cases in the theorem (i.e., for $p \in [2,m]$ and $r \in [2,\infty]$).

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G., Mansilla, Muro (2016)

Assuming a positive answer to the PIP we have

$$\begin{cases} A^m_{p,r}(n) \sim n^{\frac{1}{r}} & \text{for } (\overline{F}) : [\frac{1}{m} \leq \frac{1}{p} \leq \frac{1}{2} \wedge \frac{1}{r} \leq \frac{m}{2-m} \cdot \frac{1}{p} + \frac{m}{2m-4}], \\ A^m_{p,r}(n) \ll n^{m(\frac{1}{p}+\frac{1}{r}-\frac{1}{2})-\frac{1}{r}} & \text{for } (\overline{G}) : [\frac{1}{m} \leq \frac{1}{p} \leq \frac{1}{2} \wedge \frac{m}{2-m} \cdot \frac{1}{p} + \frac{m}{2m-4} \leq \frac{1}{r} \leq \frac{1}{2}]. \end{cases}$$

Moreover, the power of n in $(\overline{G})$ cannot be improved.

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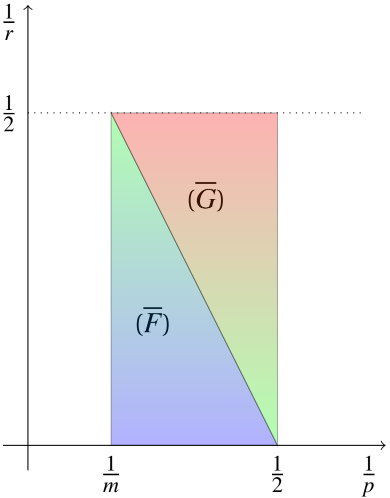

Figure: Graphical overview of the cases treated in previous theorem.

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Thanks!!!